RIGOROUS POINTWISE APPROXIMATIONS FOR INVARIANT DENSITIES OF NONUNIFORMLY EXPANDING MAPS

WAEL BAHSOUN†, CHRISTOPHER BOSE*, AND YUEJIAO DUAN‡

ABSTRACT. We use an Ulam-type discretization scheme to provide *pointwise* approximations for invariant densities of interval maps with a neutral fixed point. We prove that the approximate invariant density converges pointwise to the true density at a rate $C^* \cdot \frac{\ln m}{m}$, where C^* is a computable fixed constant and m^{-1} is the mesh size of the discretization.

1. INTRODUCTION

Ulam-type discretization schemes provide rigorous approximations for dynamical invariants. Moreover, such discretizations are easily implementable on a computer. In [18] it was shown that the original Ulam method [23] is remarkably successful in approximating isolated spectrum of transfer operators associated with piecewise expanding maps of the interval. In particular, it was shown that this method provides rigorous approximations in the L^1-norm for invariant densities of Lasota-Yorke maps (see [18] and references therein). This method has been also successful when dealing with multi-dimensional piecewise expanding maps [20], and partially successful [1] in providing rigorous approximations for certain uniformly hyperbolic systems [10, 11]. Recently, Blank [5] and Murray [21] independently succeeded in applying the pure Ulam method in a non-uniformly hyperbolic setting. They obtained approximations in the L^1-norm for invariant densities of certain non-uniformly expanding maps of the interval[2].

Although L^1 approximations provide significant information about the long-term statistics of the underlying system, they are not helpful when dealing with rare events in dynamical systems. In fact, when studying *rare events* in dynamical systems [1, 15] one often obtains probabilistic laws that depend on pointwise information from the invariant density of the system. In particular, *extreme value laws* of interval maps with a neutral fixed point depend pointwise on the invariant density of the map [13].

Statistical properties of non-uniformly expanding maps were studied by Pianigiani in [22] who first proved existence of invariant densities of such maps. Later, it

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[1]See [6] for examples where the pure Ulam method provides fake spectra for certain hyperbolic systems.

[2]In [21], in addition to proving convergence, Murray also obtained an upper bound on the rate of convergence.

was independently proved in [14, 19, 24] that such maps exhibit polynomial decay of correlations. The slow mixing behaviour that such maps exhibit has made them good testing tools for real and difficult physical problems.

The difficulties in obtaining pointwise approximations for invariant densities of interval maps with a neutral fixed point is two fold. Firstly, the transfer operator associated with such maps does not have a spectral gap in a classical Banach space. Therefore, powerful perturbation results [16][3] are not directly available in this setting. Secondly, invariant densities of such maps are not L^∞ functions. Consequently, to provide pointwise approximation of such densities, one should first measure the approximations in a 'properly weighted' $L^\infty-$norm.

In this note we use a piecewise linear Ulam-type discretization scheme to provide pointwise approximations for invariant densities of nonuniformly expanding interval maps. We prove that the approximate invariant density converges pointwise to the true density at a rate $C^* \cdot \frac{\ln m}{m}$, where C^* is a computable fixed constant and m^{-1} is the mesh size of the discretization. To overcome the spectral difficulties and the unboundedness of the densities which we discussed above, we first induce the map and obtain a uniformly piecewise, expanding and onto map. Then we perform our discretization on the induced space. After that we pull back, both the invariant density and the approximate one to the full space and measure their difference in a weighted L^∞-norm. Full details of our strategy is given in subsection 3.2.

In section 2, we recall results on uniformly piecewise expanding and onto maps. Moreover, we introduce our discretization scheme and recall results about uniform approximations for invariant densities of uniformly piecewise expanding and onto maps. In section 3, we introduce our non-uniformly expanding system, set up our strategy, and state our main results, Theorem 3.1 and Corollary 3.2. Section 4 contains technical Lemmas and the proof of Theorem 3.1.

2. Preliminaries

2.1. **A piecewise expanding system.** Let $(\Delta, \mathfrak{B}, \hat{\lambda})$ denote the measure space where Δ is an interval, $\mathfrak{B}$ is Borel σ-algebra and $\hat{\lambda}$ is normalized Lebesuge measure on Δ. Let $\hat{T} : \Delta \to \Delta$ be a measurable transformation. We assume that there exists a countable partition $\mathcal{P}$ of Δ, which consists of a sequence of intervals, $\mathcal{P} = \{I_i\}_{i=0}^{\infty}$, such that

(1) for each $i = 1, \dots, \infty$, $\hat{T}_i := \hat{T}_{|\overset{\circ}{I_i}}$ is monotone, C^2 and it extends to a C^2 function on $\bar{I}_i$;
(2) $\hat{T}_i(I_i) = \Delta$; i.e., for each $i = 1, \dots, \infty$, $\hat{T}_i$ is onto;
(3) there exists a constant $D > 0$ such that $\sup_i \sup_{x \in I_i} \frac{|\hat{T}''(x)|}{(\hat{T}'(x))^2} \leq D$;
(4) there exits a number γ such that $\frac{1}{|\hat{T}_i'|} \leq \gamma < 1$.

Let $\hat{\mathcal{L}} : L^1 \to L^1$ denote the transfer operator (Perron-Frobenius) [4, 8] associated to $\hat{T}$:

$$\hat{\mathcal{L}} f(x) = \sum_{y = \hat{T}^{-1} x} \frac{f(y)}{|\hat{T}'(y)|}.$$

[3] See also [12] for another perturbation result, which also requires a spectral gap.

Under the above assumptions, among other ergodic properties, it is well known (see for instance [7]) $\hat{T}$ admits a unique invariant density $\hat{f}$; i.e. $\hat{\mathcal{L}}\hat{f} = \hat{f}$. Moreover, $\hat{\mathcal{L}}$ admits a spectral gap when acting on the space of Lipschitz continuous functions over Δ [2][4]. We will denote by $BV(\Delta)$ the space of functions of bounded variation defined on the interval Δ. Set $||\cdot||_{BV(\Delta)} := V_\Delta + ||\cdot||_1$, where V_Δ denotes the one-dimensional variation over Δ. Then it is well known that $(BV(\Delta), ||\cdot||_{BV(\Delta)})$ is a Banach space and $\hat{\mathcal{L}}$ satisfies the following inequality (see [22] for instance): there exists a constant $C_{LY} > 0$ such that for any $f \in BV(\Delta)$, we have

$$V_\Delta \hat{\mathcal{L}} f \leq \gamma V_\Delta f + C_{LY}||f||_1. \tag{2.1}$$

Inequality (2.1) is called the Lasota-Yorke inequality.

2.2. **Markov Discretization.** We now introduce a discretization scheme which enables us to obtain rigorous uniform approximation of $\hat{f}$ the invariant density of $\hat{T}$. We use a piecewise linear approximations which was introduced by Ding and Li [9]. Let $\eta = \{c_i\}_{i=0}^m$ be a partition of I into intervals. Since uniform partitions are the first choice for numerical work, we set $c_i - c_{i-1} = \frac{1}{m}$. Everything we do can be easily modified for non-uniform partitions with only minor notational changes. Let

$$\varphi_i = \chi_{[c_{i-1}, c_i]} \text{ and } \phi_i(x) = m\int_0^x \varphi_i d\lambda.$$

Let ψ_i denote a set of **hat** functions over η:

$$\psi_0 := (1 - \phi_1)\,, \psi_m := \phi_m \text{ and for } i = 1, \dots, m-1\,, \psi_i := (\phi_i - \phi_{i+1}). \tag{2.2}$$

For $f \in L^1$, we set $I_i := [c_{i-1}, c_i]$ and

$$f_i := m\int_{I_i} f\,dx,\ i = 1, 2, \dots m,$$

the average of f over the associated partition cell. For $f \in L^1$ we set

$$Q_m f := f_1\psi_0 + \sum_{i=1}^{m-1} \frac{f_i + f_{i+1}}{2}\psi_i + f_m\psi_m$$

Obviously, the operator Q_m retains good stochastic properties; i.e.,

- for $f \geq 0$, $Q_m f \geq 0$;
- $\int Q_m f = \int f$.

We now define a piecewise linear Markov discretization of $\hat{\mathcal{L}}$ by

$$\mathbb{P}_m := Q_m \circ \hat{\mathcal{L}}. \tag{2.3}$$

Notice that $\mathbb{P}_m$ is a finite-rank Markov operator whose range is contained in the space of continuous, piecewise linear functions with respect to η. The matrix representation of $\mathbb{P}_m$ restricted to this finite-dimensional space and with respect to the basis $\{\psi_i\}$ is a (row) stochastic matrix, with entries

$$p_{ij} := m\int_{I_j} \hat{\mathcal{L}}\psi_i \geq 0.$$

[4]In [2], a Lasota-Yorke inequality was obtained for Markov interval maps with a finite partition. The proof carries over for piecewise onto maps with a countable number of branches satisfying assumptions of subsection 2.1.

By the Perron-Frobenius Theorem for stochastic matrices [17], $\mathbb{P}_m$ has a left invariant density $\hat{f}_m$; i.e.,

$$\hat{f}_m = \hat{f}_m \mathbb{P}_m.$$

The following theorem was proved in [2]:

Theorem 2.1. *There exits a computable constant $\hat{C}$ such that for any $m \in \mathbb{N}$*

$$||\hat{f} - \hat{f}_m||_\infty \le \hat{C}\frac{\ln m}{m}.$$

Remark 2.2. We recall that in [2] it was shown that the constant $\hat{C}$, which is independent of m, can be computed explicitly.

3. Pointwise Approximations for invariant densities of Maps with a neutral fixed point

3.1. **The non-uniformly expanding system.** Let $I = [0, 1]$ be the unit interval, λ be Lebesgue measure on $[0, 1]$. Let $T : I \to I$ be a piecewise smooth map. We assume that

- $T(0) = 0$ and there is a $x_0 \in (0, 1)$ such that $T_1 = T\mid_{[0,x_0)}, T_2 = T\mid_{[x_0,1]}$ and $T_1 : [0, x_0) \overset{\text{onto}}{\to} [0, 1), T_2 : [x_0, 1] \overset{\text{onto}}{\to} [0, 1]$;
- T_1 is C^1 on $[0, x_0]$, T_1 is C^2 on $(0, x_0]$ and T_2 is C^2 on $[x_0, 1]$.
- $T'(0) = 1$ and $T'(x) > 1$ for $x \in (0, x_0)$; $|T'(x)| \ge \beta > 1$ for $x \in (x_0, 1)$;
- T_1 and T_1' have the form
$$T_1(x) = x + x^{1+\alpha} + x^{1+\alpha}\delta_0(x),$$
$$T_1'(x) = 1 + (1 + \alpha)x^\alpha + x^\alpha \delta_1(x),$$
where, $0 < \alpha < 1$ and $\delta_i(x) \to 0$ as $x \to 0$ for $i = 0, 1$ with $\delta_0'(x) \ge 0$.

It is well known that T admits a unique invariant density f^* [14, 19, 22, 24] and the system $(I, T, f^* \cdot \lambda)$ exhibits a polynomial mixing rate [14, 19, 24]. Moreover, it is well known [14, 19, 24] that the T-invariant density, f^*, is not an L^∞-function. In particular, near $x = 0$, $f^*(x)$ behaves like $x^{-\alpha}$. Despite this difficulty, we will show that, for any $x \in (0, 1]$, one can obtain rigorous pointwise approximation of $f^*(x)$.

3.2. **Strategy and the statement of the main result.** We first define a Banach space which is subset of L^1, but where f^* has a finite norm. More precisely, let $\mathcal{B}$ denote the set of continuous and integrable functions on $(0, 1]$ with the norm

$$\| f \|_{\mathcal{B}} = \sup_{x\in(0,1]} |x^{1+\alpha} f(x)|.$$

When equipped with the norm $\| \cdot \|_{\mathcal{B}}$, $\mathcal{B}$ is a Banach space. The fact that $f^* \in \mathcal{B}$ follows from Lemma 3.3 of [14]. Our strategy for obtaining pointwise approximation f^* consists of the following steps:

(1) We first induce T on $\Delta \subset I$ and obtain a $\hat{T}$ which satisfies the assumptions of subsection 2.1.

(2) On Δ, we use Theorem 2.1 to say that $\hat{f}_m$, the invariant density of the discretized operator $\mathbb{P}_m := Q_m \circ \hat{\mathcal{L}}$, defined in equation (2.3), provides a uniform approximation of $\hat{f}$ the $\hat{T}$-invariant density.
(3) Then we write f^* in terms of $\hat{f}$, and define a function f_m as the 'pullback' of $\hat{f}_m$.
(4) We then use steps (2) and (3) to prove that $||f^* - f_m||_{\mathcal{B}} \le C^* \frac{\ln m}{m}$, and deduce a pointwise approximation of f^*.

3.2.1. *The induced system.* We induce T on $\Delta := [x_0, 1]$. For $n \ge 0$ we define

$$x_{n+1} = T_1^{-1}(x_n).$$

Set

$$W_0 := (x_0, 1), \text{and } W_n := (x_n, x_{n-1}),\ n \ge 1.$$

For $n \ge 1$, we define

$$Z_n := T_2^{-1}(W_{n-1}).$$

Then we define the induced map $\hat{T} : \Delta \to \Delta$ by

$$\hat{T}(x) = T^n(x) \text{ for } x \in Z_n. \tag{3.1}$$

Observe that

$$T(Z_n) = W_{n-1} \text{ and } \tau_{Z_n} = n,$$

where τ_{Z_n} is the first return time of Z_n to Δ. An example of the map T and its induced counterpart $\hat{T}$ are shown in Figures 1 and 2 respectively. It is well known (see for instance [24]) that the $\hat{T}$ defined in (3.1) satisfies the assumptions of subsection 2.1, and, by Theorem 2.1, one can obtain a rigorous *uniform* approximation of its invariant density $\hat{f}$. Moreover, by Lemma 3.3 of [3], f^*, the invariant density of T, can be written in terms of $\hat{f}$:

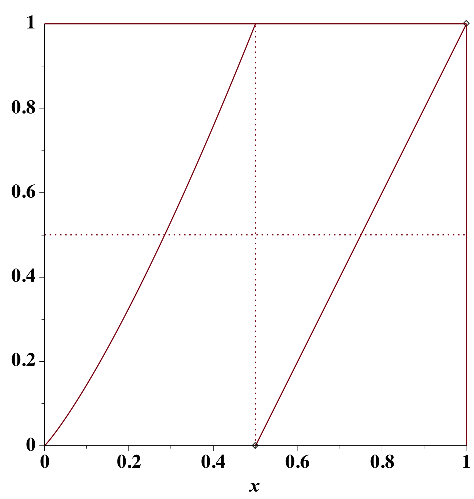

FIGURE 1. A typical example of a map T which belongs to the family defined in subsection 3.1.

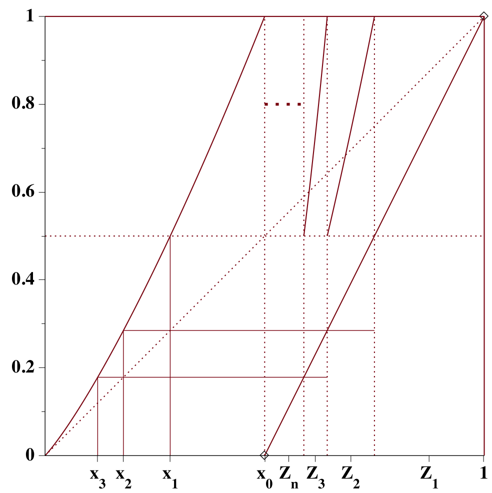

FIGURE 2. This figure shows the induced map $\hat{T}$ corresponding to the map T of Figure 1.

$$f^*(x) = \begin{cases} c_\tau \hat{f}(x) & \text{for } x \in \Delta \\ c_\tau \sum_{n=1}^{\infty} \left(\frac{\hat{f}(T_2^{-1} T_1^{-(n-1)} x)}{|DT^{(n)}(T_2^{-1} T_1^{-(n-1)} x)|} \right) & \text{for } x \in I \setminus \Delta \end{cases}, \tag{3.2}$$

where $\hat{f}$ is the $\hat{T}$-invariant density, $c_\tau^{-1} = \sum_{k=1}^{\infty} \tau_{Z_k} \hat{\mu}(Z_k)$, and $\hat{\mu} = \hat{f} \cdot \hat{\lambda}$.

3.2.2. *The approximate density and the statement of the main result.* Set

$$f_m(x) \stackrel{\text{def}}{:=} \begin{cases} c_{\tau,m} \hat{f}_m(x) & \text{for } x \in \Delta \\ c_{\tau,m} \sum_{n=1}^{\infty} \left(\frac{\hat{f}_m(T_2^{-1} T_1^{-(n-1)} x)}{|DT^{(n)}(T_2^{-1} T_1^{-(n-1)} x)|} \right) & \text{for } x \in I \setminus \Delta \end{cases}, \tag{3.3}$$

where $\hat{f}_m = \mathbb{P}_m \hat{f}_m$, and $\mathbb{P}_m$ is the Markov discretization of $\hat{\mathcal{L}}$ defined in (2.3), $c_{\tau,m}^{-1} = \sum_{k=1}^{\infty} \tau_{Z_k} \hat{\mu}_m(Z_k)$, and $\hat{\mu}_m = \hat{f}_m \cdot \hat{\lambda}$. We will show that the function f_m defined in (3.3) provides a rigorous pointwise approximation of f^*.

Theorem 3.1. *For any $m \in \mathbb{N}$ we have*

$$||f^* - f_m||_{\mathcal{B}} \le C^* \frac{\ln m}{m},$$

where

$$C^* = \hat{C} \left(1 + \frac{x_0^{1+\alpha}}{\beta} + M(1+\alpha) \right) C_4;$$

in particular, $\hat{C}$ is the computable constant of Theorem 2.1,

$$M := \frac{C_1^{1+\alpha} e^{2C_0 C_1^{2\alpha}}}{\beta},$$

$$C_0 := \frac{\alpha(1+\alpha)}{2}[1 + 2\delta_0(x_0) + \delta_0^2(x_0)], \;\; C_1 := (2[2^{\frac{1}{\alpha}} - 1])^{1/\alpha},$$

$$C_4 := 1 + C_3(\frac{C_{LY}}{1-\gamma} + \frac{1}{\Delta}), C_3 := \frac{1}{\beta} + \frac{C_2}{\beta(1-x_0)}(\alpha + \frac{2-\alpha}{1-\alpha}),$$

and

$$C_2 = \frac{1-x_0}{x_0^{1+\alpha}} 2^{1+\frac{1}{\alpha}}[2^{\frac{1}{\alpha}} - 1]^{1+\frac{1}{\alpha}}.$$

As a direct consequence of the Theorem 3.1 we obtain a pointwise approximation of f^*:

Corollary 3.2. *For any $x \in (0,1]$ we have*

$$|f^*(x) - f_m(x)| \le \frac{C^*}{x^{1+\alpha}} \frac{\ln m}{m}.$$

Proof. For $x \in (0,1]$, we have

$$|f^*(x) - f_m(x)| = \frac{1}{x^{1+\alpha}}|x^{1+\alpha}(f^*(x) - f_m(x))| \le \frac{1}{x^{1+\alpha}}||f^* - f_m||_{\mathcal{B}} \le \frac{1}{x^{1+\alpha}} C^* \frac{\ln m}{m}.$$

□

4. Proofs

4.1. **Technical lemmas.** We first introduce notation of certain functions which appear in the proof of Theorem 3.1. For $x \in I \setminus \Delta$ set:

$$g(x) := \frac{(\frac{T_1 x}{x})^{1+\alpha}}{T_1'(x)},$$

$$G_1(x) := \frac{x^{1+\alpha}}{|T_2'(T_2^{-1}x)|}; \quad \text{and for } n \ge 2, \quad G_n(x) := \frac{x^{1+\alpha}}{|DT^{(n)}(T_2^{-1}T_1^{-(n-1)}x)|}.$$

Lemma 4.1. *For $x \in I \setminus \Delta$, we have*

$$[1 + x^\alpha + x^\alpha\delta_0(x)]^{1+\alpha} \le 1 + (1+\alpha)[x^\alpha + x^\alpha\delta_0(x)] + \frac{\alpha(1+\alpha)}{2}[x^\alpha + x^\alpha\delta_0(x)]^2.$$

Proof. Let

$$\phi_1(x) := [1 + x^\alpha + x^\alpha\delta_0(x)]^{1+\alpha}$$

and

$$\phi_2(x) := 1 + (1+\alpha)[x^\alpha + x^\alpha\delta_0(x)] + \frac{\alpha(1+\alpha)}{2}[x^\alpha + x^\alpha\delta_0(x)]^2.$$

Note that $\phi_1(0) = \phi_2(0) = 1$. Therefore, to prove the lemma, it is enough to prove that $\phi_1'(x) \le \phi_2'(x)$. We have:

$$\phi_1'(x) = (1+\alpha)(1+\xi(x))^\alpha \xi'(x)$$
$$\phi_2'(x) = (1+\alpha)(1+\alpha\xi(x))\xi'(x),$$

where $\xi(x) := x^\alpha + x^\alpha\delta_0(x) \ge 0$[5]. Notice that $\xi'(x) \ge 0$. Thus, we only need to show that

$$(1+\xi(x))^\alpha \le (1+\alpha\xi(x)). \tag{4.1}$$

Indeed, (4.1) holds because $(1+\xi(0))^\alpha = (1+\alpha\xi(0)) = 1$ and

$$[(1+\xi(x))^\alpha]' = \frac{\alpha}{(1+\xi(x))^{1-\alpha}}\xi'(x) \le \alpha\xi'(x) = [1+\alpha\xi(x)]'.$$

□

[5]It is obvious that $\xi(0) = 0$ and for $x > 0$, $\xi(x) > 0$.

Lemma 4.2. *For $x \in I \setminus \Delta$, we have $g(x) \le 1 + C_0 x^{2\alpha}$, where*

$$C_0 = \frac{\alpha(1+\alpha)}{2}[1 + 2\delta_0(x_0) + \delta_0^2(x_0)].$$

Proof. Using Lemma 4.1, we have:

$$\begin{aligned}
g(x) &= \frac{(\frac{T_1 x}{x})^{1+\alpha}}{T_1'(x)} = \frac{[1 + x^\alpha + x^\alpha \delta_0(x)]^{1+\alpha}}{1 + (1+\alpha)x^\alpha + x^\alpha \delta_1(x)} \\
&\le \frac{1 + (1+\alpha)[x^\alpha + x^\alpha\delta_0(x)] + \frac{\alpha(1+\alpha)}{2}[x^\alpha + x^\alpha\delta_0(x)]^2}{1 + (1+\alpha)x^\alpha + x^\alpha\delta_1(x)} \\
&= \frac{1 + (1+\alpha)[x^\alpha + x^\alpha\delta_0(x)]}{1 + (1+\alpha)x^\alpha + x^\alpha\delta_1(x)} + \frac{\frac{\alpha(1+\alpha)}{2}[x^\alpha + x^\alpha\delta_0(x)]^2}{1 + (1+\alpha)x^\alpha + x^\alpha\delta_1(x)} \\
&\le 1 + \frac{\alpha(1+\alpha)}{2}[x^\alpha + x^\alpha\delta_0(x)]^2 \\
&= 1 + \frac{\alpha(1+\alpha)}{2}(1 + 2\delta_0(x) + \delta_0^2(x))x^{2\alpha} \le 1 + C_0 x^{2\alpha}.
\end{aligned}$$

□

Lemma 4.3. *Let $x_n = T_1^{-n} x_0$. For $n \ge 1, x_n \le C_1 n^{-\frac{1}{\alpha}}$, where $C_1 = (2[2^{\frac{1}{\alpha}} - 1])^{1/\alpha}$.*

Proof. Observe that $C_1 > 1 \ge T_1^{-1}(x_0) = x_1$. Therefore, the lemma is true for $n = 1$. Next, for $n \ge 2$, we suppose that $x_{n-1} \le C_1(n-1)^{-\frac{1}{\alpha}}$, and prove that $x_n \le C_1 n^{-\frac{1}{\alpha}}$. If it is false, that is $x_n > C_1 n^{-\frac{1}{\alpha}}$, then by our inductive statement on x_{n-1}, we have:

$$C_1(n-1)^{-\frac{1}{\alpha}} \ge x_{n-1} = T_1(x_n) > C_1 n^{-\frac{1}{\alpha}}[1 + C_1^\alpha n^{-1} + C_1^\alpha n^{-1}\delta_0(C_1 n^{-\frac{1}{\alpha}})].$$

This is equivalent to

$$n[(1 + \frac{1}{n-1})^{\frac{1}{\alpha}} - 1] > C_1^\alpha[1 + \delta_0(C_1 n^{-\frac{1}{\alpha}})].$$

By convexity of the function $z^{\frac{1}{\alpha}}$, it follows $\frac{n}{n-1}[2^{\frac{1}{\alpha}} - 1] > C_1^\alpha[1 + \delta_0(C_1 n^{-\frac{1}{\alpha}})]$, that is

$$C_1^\alpha < \frac{n}{n-1}[2^{\frac{1}{\alpha}} - 1]/[1 + \delta_0(C_1 n^{-\frac{1}{\alpha}})] < 2[2^{\frac{1}{\alpha}} - 1] = C_1^\alpha.$$

A contradiction. Therefore, $x_n \le C_1 n^{-\frac{1}{\alpha}}$, and this completes the proof of the lemma. □

Lemma 4.4. *For $x \in I \setminus \Delta$, we have*

$$G_1(x) \le \frac{x_0^{1+\alpha}}{\beta}.$$

and for $n \ge 2$,

$$G_n(x) \le M(n-1)^{-(1+\frac{1}{\alpha})},$$

where $M = \frac{C_1^{1+\alpha} e^{2C_0 C_1^{2\alpha}}}{\beta}$.

Proof. For $n=1$, it is easy to see that

$$G_1(x) \le \frac{x_0^{1+\alpha}}{\beta}.$$

For $n \ge 2$, we have

$$\begin{aligned}
G_n(x) &= \frac{x^{1+\alpha}}{|DT^{(n)}(T_2^{-1}T_1^{-(n-1)}x)|} \\
&= \frac{x^{1+\alpha}}{|D(T_1 \circ T_1 \circ \cdots \circ T_1 \circ T_2)(T_2^{-1}T_1^{-(n-1)}x)|} \\
&= \frac{x^{1+\alpha}}{T_1'(T_1^{-1}x)\cdot T_1'(T_1^{-2}x)\cdots T_1'(T_1^{-(n-1)}x)\cdot|T_2'(T_2^{-1}T_1^{-(n-1)}x)|} \\
&= \frac{(\frac{x}{T_1^{-1}x})^{1+\alpha}}{T_1'(T_1^{-1}x)}\cdot\frac{(\frac{T_1^{-1}x}{T_1^{-2}x})^{1+\alpha}}{T_1'(T_1^{-2}x)}\cdots\frac{(\frac{T_1^{-(n-2)}x}{T_1^{-(n-1)}x})^{1+\alpha}}{T_1'(T_1^{-(n-1)}x)}\cdot\frac{(T_1^{-(n-1)}x)^{1+\alpha}}{|T_2'(T_2^{-1}T_1^{-(n-1)}x)|} \\
&= g(T_1^{-1}x)\cdot g(T_1^{-2}x)\cdots g(T_1^{-(n-1)}x)\cdot\frac{(T_1^{-(n-1)}x)^{1+\alpha}}{|T_2'(T_2^{-1}T_1^{-(n-1)}x)|} \\
&\le g(T_1^{-1}x)\cdot g(T_1^{-2}x)\cdots g(T_1^{-(n-1)}x)\cdot\frac{(T_1^{-(n-1)}x)^{1+\alpha}}{\beta}.
\end{aligned} \tag{4.2}$$

By Lemmas 4.2 and 4.3, for any $k \ge 1, x \in [0, x_0)$, we have

$$\begin{aligned}
g(T_1^{-k}(x)) &\le 1 + C_0(T_1^{-k}(x))^{2\alpha} \le 1 + C_0(T_1^{-k}(x_0))^{2\alpha} \\
&= 1 + C_0(x_k)^{2\alpha} \le 1 + C_0C_1^{2\alpha}k^{-2}.
\end{aligned} \tag{4.3}$$

Therefore, using (4.2) and (4.3), for $n \ge 2$, we obtain:

$$\begin{aligned}
G_n(x) &= \prod_{k=1}^{n-1} g(T_1^{-k}(x))\cdot\frac{(T_1^{-(n-1)}(x))^{1+\alpha}}{\beta} \\
&\le \prod_{k=1}^{n-1}(1 + C_0C_1^{2\alpha}k^{-2})\cdot\frac{C_1^{1+\alpha}(n-1)^{-(1+\frac{1}{\alpha})}}{\beta} \\
&= \exp\{\sum_{k=1}^{n-1}\ln(1 + C_0C_1^{2\alpha}k^{-2})\}\cdot\frac{C_1^{1+\alpha}(n-1)^{-(1+\frac{1}{\alpha})}}{\beta} \\
&\le \exp\{\sum_{k=1}^{n-1} C_0C_1^{2\alpha}k^{-2}\}\cdot\frac{C_1^{1+\alpha}(n-1)^{-(1+\frac{1}{\alpha})}}{\beta} \\
&\le \exp\{C_0C_1^{2\alpha}(2 - \frac{1}{n-1})\}\cdot\frac{C_1^{1+\alpha}(n-1)^{-(1+\frac{1}{\alpha})}}{\beta} \\
&\le M(n-1)^{-(1+\frac{1}{\alpha})}.
\end{aligned}$$

□

Lemma 4.5.

$$\sum_{n=1}^{\infty} n\cdot\hat{\lambda}(Z_n) \le C_3,$$

where $C_3 = \frac{1}{\beta} + \frac{C_2}{\beta(1-x_0)}(\alpha + \frac{2-\alpha}{1-\alpha})$ *and* $C_2 = \frac{1-x_0}{x_0^{1+\alpha}}2^{1+\frac{1}{\alpha}}[2^{\frac{1}{\alpha}} - 1]^{1+\frac{1}{\alpha}}$.

Proof. By Lemma 4.3, we have $\lambda(W_n) = x_{n-1} - x_n = T_1(x_n) - x_n = \frac{1-x_0}{x_0^{1+\alpha}} x_n^{1+\alpha} \leq \frac{1-x_0}{x_0^{1+\alpha}} C_1^{1+\alpha} n^{-(1+\frac{1}{\alpha})} = C_2 n^{-(1+\frac{1}{\alpha})}$. Since $T_2(Z_n) = W_{n-1}$, we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} n \cdot \lambda(Z_n) &\leq \lambda(Z_1) + \sum_{n=2}^{\infty} n \cdot \frac{\lambda(W_{n-1})}{\beta} \\
&\leq \frac{1-x_0}{\beta} + \sum_{n=2}^{\infty} \frac{n(x_{n-2} - x_{n-1})}{\beta} \\
&= \frac{1-x_0}{\beta} + \sum_{n=1}^{\infty} \frac{(n+1)(x_{n-1} - x_n)}{\beta} \\
&= \frac{1-x_0}{\beta} + \sum_{n=1}^{\infty} \frac{n(x_{n-1} - x_n)}{\beta} + \sum_{n=1}^{\infty} \frac{(x_{n-1} - x_n)}{\beta} \\
&\leq \frac{1-x_0}{\beta} + \sum_{n=1}^{\infty} \frac{C_2}{\beta} n^{-\frac{1}{\alpha}} + \sum_{n=1}^{\infty} \frac{C_2}{\beta} n^{-(1+\frac{1}{\alpha})} \\
&\leq \frac{1-x_0}{\beta} + \frac{C_2}{\beta}(1 + \int_1^{\infty} x^{-\frac{1}{\alpha}} dx) + \frac{C_2}{\beta}(1 + \int_1^{\infty} x^{-(1+\frac{1}{\alpha})} dx) \\
&= \frac{1-x_0}{\beta} + \frac{C_2}{\beta}(\alpha + \frac{2-\alpha}{1-\alpha}) = (1-x_0) \cdot C_3.
\end{aligned}
$$

This completes the proof of the lemma since $\hat{\lambda}(\cdot) = \frac{\lambda(\cdot)}{1-x_0}$. □

Lemma 4.6. *We have*

$$|c_{\tau,m} - c_\tau| \leq C_3 \cdot \hat{C} \frac{\ln m}{m}.$$

Proof. Using the fact that $c_\tau \leq 1$, $c_{\tau,m} \leq 1$ and Theorem 2.1, we have

$$
\begin{aligned}
|c_{\tau,m} - c_\tau| &\leq \left| \frac{1}{\sum_{k=1}^{\infty} \tau_{Z_k} \hat{\mu}_m(Z_k)} - \frac{1}{\sum_{k=1}^{\infty} \tau_{Z_k} \hat{\mu}(Z_k)} \right| \\
&= \left| \frac{\sum_{k=1}^{\infty} k[\hat{\mu}(Z_k) - \hat{\mu}_m(Z_k)]}{\sum_{k=1}^{\infty} \tau_{Z_k} \hat{\mu}_m(Z_k) \cdot \sum_{k=1}^{\infty} \tau_{Z_k} \hat{\mu}(Z_k)} \right| \\
&\leq \left(\sum_{k=1}^{\infty} k \int_{Z_k} |\hat{f} - \hat{f}_m| d\hat{\lambda} \right) \\
&\leq ||\hat{f} - \hat{f}_m||_\infty \left(\sum_{k=1}^{\infty} k \hat{\lambda}(Z_k) \right) \\
&\leq \hat{C} \frac{\ln m}{m} \cdot C_3.
\end{aligned}
$$

In the last estimate, we have used Lemma 4.5. □

We now have all our tools ready to prove Theorem 3.1.

Proof. (**of Theorem 3.1**) Using (3.2) and (3.3), we have

(4.4)

$$||f^* - f_m||_{\mathcal{B}} = \sup_{x\in(0,1]} |x^{1+\alpha}(f^*(x) - f_m(x))|$$
$$\le \sup_{x\in I\setminus\Delta} |x^{1+\alpha}(f^*(x) - f_m(x))| + \sup_{x\in\Delta} |x^{1+\alpha}(f^*(x) - f_m(x))|$$
$$= \sup_{x\in I\setminus\Delta} |\sum_{n=1}^{\infty} \frac{x^{1+\alpha}}{DT^{(n)}(T_2^{-1}T_1^{-(n-1)}x)}(c_\tau \hat{f}(T_2^{-1}T_1^{-(n-1)}x) - c_{\tau,m}\hat{f}_m(T_2^{-1}T_1^{-(n-1)}x))|$$
$$+ \sup_{x\in\Delta} |x^{1+\alpha}(c_\tau \hat{f}(x) - c_{\tau,m}\hat{f}_m(x))|.$$

Notice that for $x \in I \setminus \Delta$, and $n \ge 1$, $z_n := T_2^{-1}T_1^{-(n-1)}x \in \Delta$. Then using the fact that $c_\tau \le 1$, $c_{\tau,m} \le 1$, Theorem 2.1, Lemma 4.6, and (4.4), we obtain:

(4.5)

$$||f^* - f_m||_{\mathcal{B}} \le \sup_{x\in I\setminus\Delta} |\sum_{n=1}^{\infty} \frac{x^{1+\alpha}}{DT^{(n)}(T_2^{-1}T_1^{-(n-1)}x)}| \cdot \sup_{z_n\in\Delta} |(c_\tau \hat{f}(z_n) - c_{\tau,m}\hat{f}_m(z_n)|$$
$$+ \sup_{x\in\Delta} |c_\tau \hat{f}(x) - c_{\tau,m}\hat{f}_m(x)|$$
$$\le \sup_{x\in I\setminus\Delta} |\sum_{n=1}^{\infty} \frac{x^{1+\alpha}}{DT^{(n)}(T_2^{-1}T_1^{-(n-1)}x)}| \times$$
$$\left(\sup_{z_n\in\Delta} |\hat{f}(z_n) - \hat{f}_m(z_n)| + |c_\tau - c_{\tau,m}| \sup_{z_n\in\Delta} |\hat{f}(z_n)|\right)$$
$$+ \sup_{x\in\Delta} |\hat{f}(x) - \hat{f}_m(x)| + |c_\tau - c_{\tau,m}| \sup_{x\in\Delta} |\hat{f}(x)|$$
$$\le \hat{C}\frac{\ln m}{m}\left(\sup_{x\in I\setminus\Delta} \sum_{n=1}^{\infty} |G_n(x)|(1 + C_3 \sup_{z_n\in\Delta} |\hat{f}(z_n)|) + (1 + C_3 \sup_{x\in\Delta} |\hat{f}(x)|)\right).$$

Since $\hat{f} \in BV(\Delta)$, we have $\sup_{x\in\Delta} |\hat{f}(x)| \le V_\Delta \hat{f} + \frac{1}{1-x_0}||\hat{f}||_{1,\Delta}$. Therefore, using the Lasota-Yorke inequality (2.1), we obtain $\sup_{x\in\Delta} |\hat{f}(x)| \le \frac{C_{LY}}{1-\gamma} + \frac{1}{1-x_0}$. Using Lemma 4.4 and (4.5), we obtain:

$$\begin{aligned}
||f^* - f_m||_{\mathcal{B}} &\le C_4\hat{C}\frac{\ln m}{m}\left(1 + \frac{x_0^{1+\alpha}}{\beta} + \sum_{n=2}^{\infty} M(n-1)^{-(1+\frac{1}{\alpha})}\right) \\
&= C_4\hat{C}\frac{\ln m}{m}\left(1 + \frac{x_0^{1+\alpha}}{\beta} + M\sum_{n=1}^{\infty} n^{-(1+\frac{1}{\alpha})}\right) \\
&\le C_4\hat{C}\left(1 + \frac{x_0^{1+\alpha}}{\beta} + M(1+\alpha)\right) \cdot \frac{\ln m}{m}.
\end{aligned}$$

□

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Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire, LE11 3TU, UK

E-mail address: † W.Bahsoun@lboro.ac.uk, ‡ Y.Duan@lboro.ac.uk

Department of Mathematics and Statistics, University of Victoria, PO BOX 3045 STN CSC, Victoria, B.C., V8W 3R4, Canada

E-mail address: ∗ cbose@uvic.ca